Journal of Engineering Mathematics, Vol. 6, No. 1, January 1972
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

Comment

ON "THE SHALLOW WATER EQUATIONS" BY M. SHINBROT

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(Received March 23, 1971)

SUMMARY
The paper [1] above describes two steady shallow water flows which differ from the known flows of this type. It is suggested that the conditions imposed in obtaining these solutions are not consistent.

Steady Flow on Shallow Water of Constant Depth

Lamb [2], §§252, 3 derives solutions for two steady shallow water flows, namely, the solitary wave and the periodic cnoidal wave train. The relationship between these two solutions and the hydraulic jump is investigated by Benjamin and Lighthill [3]. Shinbrot [1] finds a periodic solution which, although cnoidal, differs from the family of cnoidal wave trains described by Lamb [2], and a laminar jump which differs from the hydraulic jump of Benjamin and Lighthill [3].

The notation below is the same as that used by Shinbrot [1], sec. 4, with the addition of $\bar{u}(x)$ for the mean horizontal velocity over vertical cross-sections of the flow. Consider now the two equations in [1] labelled (4.1), both of which may be derived directly from first principles. The first equation is an exact statement of Bernoulli's equation for the surface streamline of a steady inviscid flow, namely,

$$g\eta + \tfrac{1}{2}(u^2 + v^2) = \text{constant},$$

where $v = u\eta_x$ for kinematic reasons. The second equation is an approximate statement of the conservation of mass for a steady flow, the exact equation being

$$\bar{u}\eta = \text{constant}.$$

The two exact equations are therefore

$$g\eta + \tfrac{1}{2}u^2(1+\eta_x^2) = \text{constant}, \tag{1}$$

$$u\eta + (\bar{u}-u)\eta = \text{constant}. \tag{2}$$

Shinbrot [1] assumes, in effect, that the neglect of the second term in equation (2) for shallow water flow does not require the neglect of the term in η_x^2 in equation (1), or in other words, that there exists a class of shallow water flows for which the condition

$$|\bar{u}-u| \ll |u|, \tag{3}$$

does not include the condition

$$\eta_x^2 \ll 1. \tag{4}$$

Let l be a horizontal length scale for the flow, h be a scale for the flow depth, and α be a measure of the amplitude of the surface displacement. Shinbrot [1] reduces condition (3) to a condition labelled (3.4), namely

$$ah/s^2 \ll 1, \tag{5}$$

where a is a typical acceleration and s a typical speed in the flow. A typical acceleration term in a steady flow is uu_x, which is $O(s^2/l)$, so condition (5) becomes

$$h/l \ll 1 . \tag{6}$$

Hence condition (3) states that a quantity $O(h/l)$ is neglected compared with 1, whereas condition (4) states that a quantity $O(\alpha^2/l^2)$ is neglected compared with 1. Also $\alpha^2/l^2 \ll h/l$, since $\alpha < h$. It is inconsistent therefore to impose condition (3) without also imposing condition (4), since η_x^2 is always very much smaller than the quantity neglected by condition (3). A further difficulty is that the horizontal length scale in each of the two solutions found by Shinbrot [1] is of the order of h, that is,

$$h/l \sim 1 ,$$

which is inconsistent with condition (6).

Consider now the shallow water flow consisting of a steady wave motion on an otherwise uniform inviscid flow. This is the situation examined by Lamb [2], §§252, 3, from which it may be deduced that

$$\bar{u} - u = \tfrac{1}{3}h^2 u_{xx} + O(\bar{u}h^4/l^4) . \tag{7}$$

The two solutions in Lamb [2] are obtained by neglecting the term in η_x^2 in equation (1), and by replacing $\bar{u} - u$ in equation (2) with the first term in equation (7). These approximations are consistent because both solutions satisfy

$$al^2/h^3 \sim 1 .$$

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